Journal of Applied Mechanics and Technical Physics, Vol. 44, No. 4, pp. 482–488, 2003

NUMERICAL INVESTIGATION OF THE FLUTTER OF A RECTANGULAR PLATE

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UDC 539.3:534.1

The flutter of a rectangular plate with an arbitrary direction of the velocity vector relative to the plate side is studied. A numerical no-saturation algorithm is constructed to solve the eigenvalue problem. Calculation results for the critical flutter velocity and corresponding eigenmodes are given.

Key words: *flutter, rectangular plate, numerical no-saturation algorithm, eigenvalue problem.*

The panel flutter of rectangular plates, as a rule, has been studied in a partial formulation [1] with the flow velocity vector parallel to one of the plate sides (see also [2]). In papers [3, 4], concerned with the plate flutter under the action of shearing and normal forces, there is no substantiation of the derivation of the equations used. In [5, 6], the problem of the panel flutter of shallow shells and plates is considered in general form using the plan section law) of supersonic aerodynamics and formulations of new problems are given. Conventional methods of numerical solution of these problems (difference methods, finite-element methods — saturation methods [7]) appear to be ineffective [8], the accuracy of the Bubnov–Galerkin method in the solution of these problems has not been studied.

In the present paper, the numerical-analytical no-saturation algorithm [7, 8] developed for the plate flutter with an arbitrary smooth contour is extended to the case of a rectangular plate with an arbitrarily oriented flow velocity vector. Calculation results are compared with results obtained using the Bubnov–Galerkin method in an eight-term approximation.

1. Formulation of the Problem. The problem of the flutter of a rectangular plate occupying a region $K = \{(x, y)\colon\ |x| \leqslant 1,\ |y| \leqslant b\}$ reduces to determination of the eigenfunctions of the system [6, 8]:

$$D\Delta^2\varphi - k\boldsymbol{n}v \operatorname{grad}\varphi = \lambda\varphi, \qquad \varphi = \varphi(x, y), \quad x, y \in K; \tag{1.1}$$

$$\varphi\Big|_{\partial K} = 0; \tag{1.2}$$

$$\frac{\partial\varphi}{\partial x}\Big|_{|x|=1} = 0, \qquad \frac{\partial\varphi}{\partial y}\Big|_{|x|=b} = 0; \tag{1.3}$$

$$\frac{\partial^2\varphi}{\partial x^2}\Big|_{|x|=1} = 0, \qquad \frac{\partial^2\varphi}{\partial y^2}\Big|_{|x|=b} = 0. \tag{1.4}$$

Here $D = Eh^3/(12(1-\mu^2)p_0a^3)$ is the dimensionless rigidity of the plate, E and μ are Young's modulus and Poisson's constant of the plate material, h is the thickness, a is half-length in the x direction, k is the polytropic exponent of the gas, p_0 is the undisturbed pressure, $v = |\boldsymbol{V}|$, $\boldsymbol{V} = \{v\cos\theta, v\sin\theta\}$ is the flow velocity normalized by the velocity of sound in it, the vector $\boldsymbol{n} = \{\cos\theta, \sin\theta\}$ defines the direction $\boldsymbol{V}$, and $\varphi = \varphi(x, y)$ is the deflection amplitude $w = \varphi\exp(\omega t)$; therefore,

$$\lambda = -\rho h\omega^2 - k\omega, \tag{1.5}$$

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 0021-8944/03/4404-0482 $25.00

where ρ is the dimensionless density of the plate material normalized by the parameter p_0/c_0^2 and the thickness h is normalized by a. Boundary conditions (1.2) and (1.3) imply that the plate is clamped at its edge and boundary conditions (1.2) and (1.4) imply free bearing.

The plate oscillations are stable at $\mathrm{Re}\,\omega < 0$ and unstable at $\mathrm{Re}\,\omega > 0$. In the complex plane $\lambda = \alpha + \beta i$, the regions of stable and unstable oscillations according to (1.5) are separated by the stability parabola [1] $F(\alpha, \beta) = k^2\alpha - \rho h\beta^2 = 0$. Since $\alpha = \alpha(v, \theta)$, $\beta = \beta(v, \theta)$, the equation $F(\alpha(v, \theta), \beta(v, \theta)) \equiv f(v, \theta) = 0$ in the plane of the parameters v and θ defines the neutral curve separating the region of their subcritical values.

The following common properties of the eigenvalues of problem (1.1)–(1.4) are known [1, 8, 9]: 1) $\mathrm{Re}\,\lambda > 0$; 2) the oscillations corresponding to the real values of λ are unstable; 3) for fixed θ, the eigenvalues sequentially pass into the complex region as v increases; for a specified value of v, the number of complex values λ is finite. According to this, the following scheme of investigation is adopted: (a) a discrete analog is put in correspondence to problem (1.1)–(1.4); (b) for fixed θ, the critical velocity is determined from the first eigenvalue; (c) with the critical velocity obtained, a stability analysis is performed for other complex eigenvalues; (d) if a complex value of λ is found outside the stability parabola, the critical velocity is evaluated from this eigenvalue; (e) the smallest value is chosen from all the critical velocities thus obtained.

2. Discretization. We shall construct a discrete Laplacian H with boundary condition (1.2) using the procedure described in [10].

In the plane (x, y), we choose a grid consisting of nodes

$$x_q = \cos\left((2q-1)\pi/(2n)\right), \qquad q = 1, 2, \ldots, n; \tag{2.1}$$

$$y_\mu = b\cos\left((2\mu-1)\pi/(2m)\right), \qquad \mu = 1, 2, \ldots, m. \tag{2.2}$$

Let A be the matrix of the discrete operator corresponding to the differential operator $\partial^2\varphi/\partial x^2$ with the boundary condition $\varphi(-1) = \varphi(1) = 0$ on grid (2.1) and let B be the matrix of the discrete operator corresponding to the differential operator $\partial^2\varphi/\partial y^2$ with the boundary condition $\varphi(-b) = \varphi(b) = 0$ on grid (2.2). Then, the discrete Laplacian becomes

$$H = I_m \otimes A + B \otimes I_n, \tag{2.3}$$

where I_n and I_m are unit matrices of size $n \times n$ and $m \times m$; the "$\otimes$" sign denotes the Kronecker matrix product. The eigenvector of the matrix H has the form $u = r \otimes s$, where s is the eigenvector of the matrix A and r is the eigenvector of the matrix B. In this case, the grid nodes are numbered first on x and then on y (from right to left and from bottom to top). One can state that matrix (2.3) inherits the variable separation property of the differential Laplacian.

Discretization of the operator $\partial^2\varphi/\partial x^2$ with the boundary condition $\varphi(a) = \varphi(b) = 0$ is performed as follows: (a) on grid (2.1) ($a = -1$, $b = 1$) or (2.2) ($a = -b$, $b = b$), the interpolation Lagrange formula satisfying the boundary conditions is written; (b) the second derivatives at the grid nodes are evaluated by differentiating the interpolation formula. As a result, we have

$$D_{ij} = \frac{2}{b-a}\,\frac{2}{k\sin^2\psi_j}\sum_{q=0}^{k-1}\cos(q\psi_j)\Big[(2+q^2)\cos(q\psi_i) + 3q\cos\psi_i + 3q\cos\psi_i\,\frac{\sin(q\psi_i)}{\sin\psi_i}\Big],$$

$$\psi_j = (2j-1)\pi/(2k), \qquad i, j = 1, 2, \ldots, k.$$

Here $k = n$, $a = -1$, and $b = 1$ for the matrix A and $k = m$, $a = -b$, and $b = b$ for the matrix B.

Discretization of the derivatives $\partial\varphi/\partial x$ and $\partial\varphi/\partial y$ is performed similarly. The interpolation Lagrange polynomial is written on the corresponding grid (2.1) or (2.2), and the values of the derivatives at the grid nodes are obtained by differentiating this interpolation formula. As a result, we have obtain the differentiation matrix

$$D_{s\mu} = \frac{4}{k(b-a)}\sum_{q=0}^{k-1}\frac{q\cos(q\psi_\mu)\sin(q\psi_s)}{\sin\psi_s}, \qquad \psi_s = \frac{(2s-1)\pi}{2k}, \quad s, \mu = 1, 2, \ldots, k.$$

For $k = n$, $a = -1$, and $b = 1$, we obtain the matrix D^x of differentiation with respect to x; for $k = m$, $a = -b$, and $b = b$, we have the matrix D^y of differentiation with respect to y. To obtain the derivatives of the function φ at the grid nodes, it is necessary to multiply the matrix D by the vector of the values of the function φ at the grid nodes. A consequence of boundary conditions (1.2) and (1.4) is the condition

$$\Delta\varphi\Big|_{\partial K} = 0. \tag{2.4}$$

In this case, the matrix of the biharmonic operator with boundary conditions (1.2) and (2.4) is H^2 because the matrix H^2 has the same eigenvectors as the matrix H and the eigenvalues λ_i^2, $i = 1, 2, \ldots, N$ [λ_i are the eigenvalues of the matrix H of size $N \times N$ ($N = mn$)].

Let us consider discretization of Eq. (1.1) with boundary conditions (1.2) and (1.3), i.e., a plate clamped along the contour.

For the function $\varphi = \varphi(x, y)$ in the rectangle, we write the interpolation formula

$$\varphi(x,y) = \sum_{j=1}^{n}\sum_{i=1}^{m} M_{i0}(z)L_{j0}(x)\varphi(x_j, y_i), \qquad y = bz, \quad z \in [-1,1],\ x \in [-1,1], \tag{2.5}$$

where

$$L_{j0}(x) = l(x)/(l'(x_j)(x - x_j)), \quad l(x) = (x^2-1)^2 T_n(x), \quad T_n(x) = \cos n \arccos x,$$

$$x_j = \cos\vartheta_j, \qquad \vartheta_j = (2j-1)\pi/(2n), \qquad j = 1, 2, \ldots, n,$$

$$M_{i0}(z) = M(z)/(M'(z_i)(z - z_i)), \qquad M(z) = (z^2-1)^2 T_m(z),$$

$$z_i = \cos\vartheta_i, \qquad \vartheta_i = (2i-1)\pi/(2m), \qquad i = 1, 2, \ldots, m.$$

The interpolation formula (2.5) satisfies the clamped boundary conditions. To obtain the matrix of the discrete biharmonic operator H, it is necessary to apply the biharmonic operator to the interpolation formula (2.5); it is necessary to differentiate formula (2.5) with respect to x and y four times. As a result, we obtain an asymmetric matrix H of size $N \times N$ ($N = mn$). Let us number the nodes in the rectangle (x_j, y_i) first along y and then along x, i.e., from top to bottom and from right to left. As a result, the expression $\Delta^2\varphi$ is approximated by the expression $H\varphi$ [φ is the vector of the values of the function $\varphi = \varphi(x, y)$ at the grid nodes]. We note that the matrix H is asymmetric although the biharmonic operator considered is self-conjugate. Consequently, the matrix H can have complex eigenvalues. In stability problems, the presence of complex eigenvalues (due to discretization errors) for the discrete biharmonic operator is undesirable. Therefore, the approach used should be modified. Instead of the matrix H, we considered the matrix $(H + H^*)/2$. This approach can be explained as follows. The original problem is self-conjugate (biharmonic equation with a clamped boundary condition) but discretization results in an asymmetric matrix H. Let us write H as

$$H = (H + H^*)/2 + (H - H^*)/2$$

and consider the asymmetric part as the discretization error. The perturbation thus introduced into the eigenvalues of the matrix H depends on how far the resolvents of the matrices H and $(H + H^*)/2$ are close in the part of the complex plane of interest for the flow stability analysis. This perturbation can be estimated theoretically using the scheme described in [11]. Results of numerical evaluation of this perturbation are given below.

For $b = 1$, a matrix H of size 361×361 ($361 = 19 \times 19$) has the first eigenvalue $\sqrt{\lambda_1}/\pi^2 = 2.4902$. This value was compared with the one obtained in calculations [12] $\sqrt{\lambda_1^*}/\pi^2 = 2.489$. The matrix $(H+H^*)/2$ has an eigenvalue $\sqrt{\lambda_1'}/\pi^2 = 2.3961$. Thus, the perturbation introduced into the eigenvalues of the matrix H by symmetrization is acceptable.

Discretization of $\operatorname{grad}\varphi$ in the boundary-value problem (1.1)–(1.3) was performed similarly.

3. Results of Numerical Calculations. We consider calculation results for a simply supported plate using the following values of mechanical parameters: $p_0 = 1.0126 \cdot 10^5$ Pa, $\rho_0 = 1.2928$ kg/m^3, $\mu = 0.33$, $k = 1.4$, $E = 6.86 \cdot 10^{10}$ Pa, and $\rho = 2.7 \cdot 10^3$ kg/m^3. The relative size b, thickness h, flow rate v, and angle θ were varied.

Test calculations were performed for a square plate ($b = 1$, $h = 0.003$). The following results were obtained: $v_{\rm cr}(0) = v_{\rm cr}(\pi/2) = 0.2103$ and $v_{\rm cr}(\pi/4) = 0.2001$; in all cases, $v_{\rm cr}$ was determined from the first eigenvalue. Curves of $\operatorname{Re}\varphi(x, 0)$ and $\operatorname{Re}\varphi(0, y)$ coincide.

Calculations results for a plate with dimensions $b = 0.5$ and $h = 0.003$ are given in Table 1 (the eigenvalue number is given in parentheses). We note the following circumstance, which can be important in numerical analysis of the flutter problem: the values $|\lambda_1| = 1.56658$ and $|\lambda_2| = 1.56660$ ($\theta = 0$) are close, but $\lambda_1 > 0$ is real and does not generate unstable oscillation modes. Values of $v_{\rm cr}^*$ obtained by the Bubnov–Galerkin method in an eight-term approximation are given in the third column of Table 1. From Table 1, it follows that this method gives understated results.

TABLE 1

θ	$v_{\rm cr}$	$v^*_{\rm cr}$
0	0.3546 (1)	0.3042
$\pi/8$	0.3737 (1)	0.3307
$\pi/4$	0.4346 (1)	—
$5\pi/16$	0.4801 (1)	0.4207
$3\pi/8$	0.5235 (1)	—
$15\pi/32$	0.5275 (2)	0.4022
$\pi/2$	0.5257 (2)	0.4121

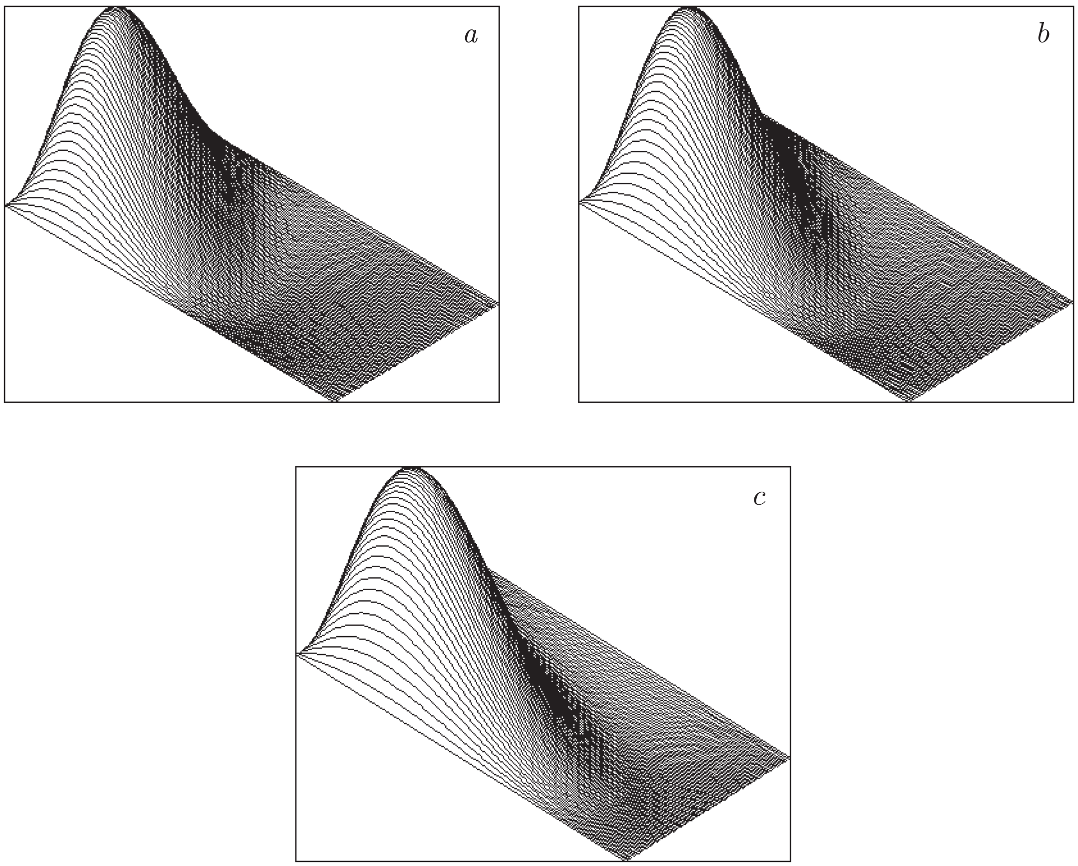

Fig. 1

The calculation results given in Table 1 lead to the following conclusions: a) the critical velocity increases abruptly in the angle range $\theta \in (\pi/4, 3\pi/8)$ and varies smoothly for other values of the angles; b) the maximum of the critical flutter velocity is near the point $\theta = 15\pi/32$ (so-called stabilization effect of plate oscillations with respect to fluctuations in the velocity vector direction in the neighborhood of $\theta = \pi/2$). We note that in [13], this effect was detected when solving the strip flutter problem. Figure 1a–c shows the real parts of the eigenfunctions for $\theta = \pi/4$, $5\pi/16$, and $3\pi/8$ and $v = v_{\rm cr} = 0.4346$, 0.4801, and 0.5235, respectively

Calculations were performed for an elongated plate with dimensions $b = 0.25$ and $h = 0.0015$ (ratio of the thickness to the smaller side of the plate is the same as in the previous calculation). The calculation results are given in Table 2. Figure 2a–e shows the real parts of the eigenfunction for $\theta = 0$, $\pi/4$, $5\pi/16$, $3\pi/8$, and $7\pi/16$ and $v = v_{\rm cr} = 0.2655$, 0.3541, 0.4014, 0.4803, and 0.4912, respectively. It is evident that for the values of θ for which the critical velocity increases most sharply, the oscillation mode changes. Consequently, for these values of θ, the plate is most sensitive to variations in the flow velocity and direction.

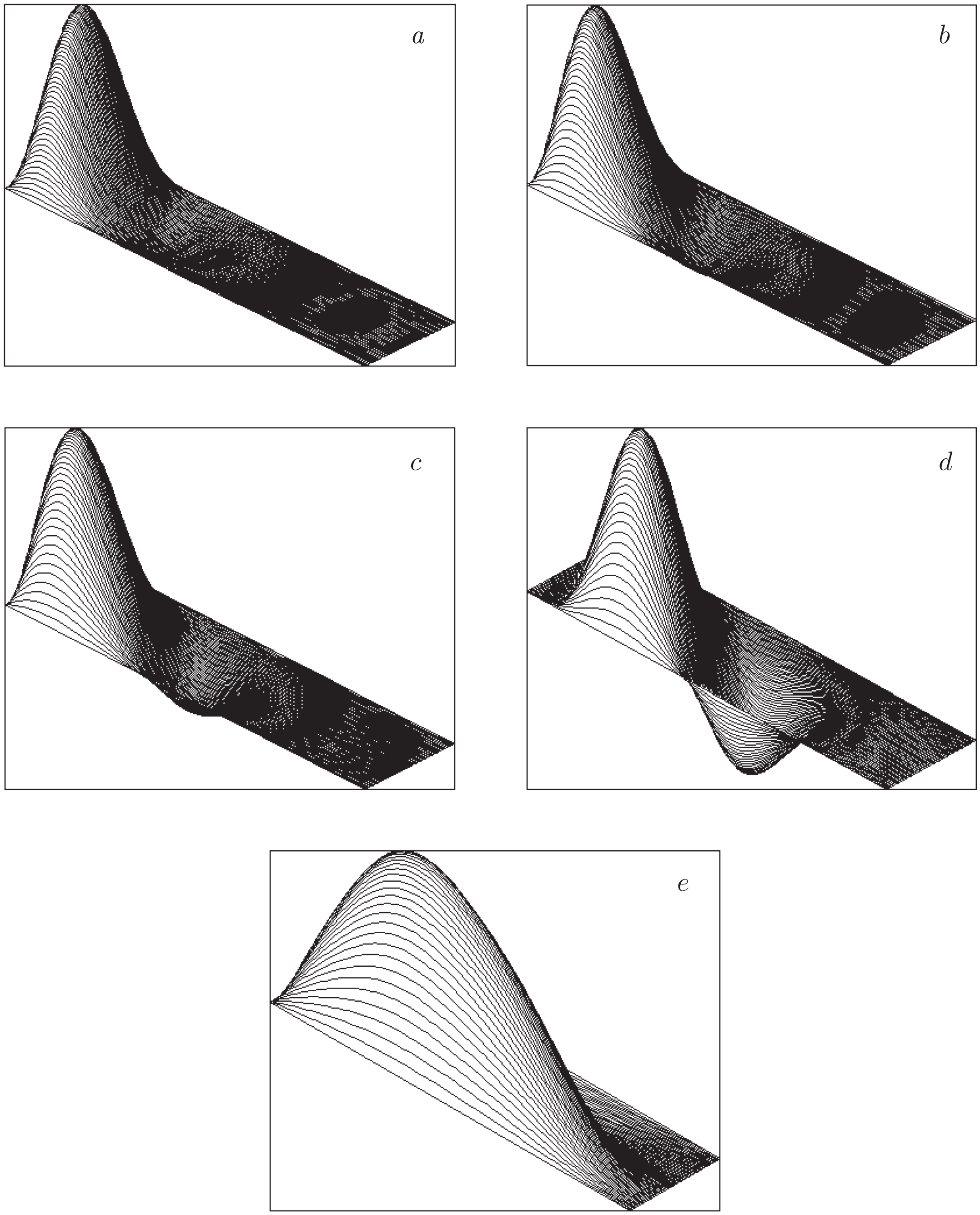

Fig. 2

TABLE 2

θ	$v_{\rm cr}$	θ	$v_{\rm cr}$
0	0.2655 (3)	$3\pi/8$	0.4803 (1)
$\pi/8$	0.2832 (3)	$7\pi/16$	0.4912 (2)
$\pi/4$	0.3453 (1)	$15\pi/32$	0.4867 (3)
$5\pi/16$	0.4014 (1)	$\pi/2$	0.4851 (4)

TABLE 3

h	$v_{\rm cr}$		
	Data [14]	Grid 9×9	Grid 19×19
$5.0000 \cdot 10^{-3}$	1.0000	1.0615 (1)	1.0615 (1)
$6.3091 \cdot 10^{-3}$	2.0029	2.0991 (1)	2.0991 (1)
$7.2202 \cdot 10^{-3}$	3.0029	3.1325 (1)	3.1324 (1)
$7.9365 \cdot 10^{-3}$	4.0029	4.1523 (1)	4.1523 (1)
$8.5470 \cdot 10^{-3}$	5.0059	5.1806 (1)	5.1805 (1)
$9.0909 \cdot 10^{-3}$	6.0059	6.2296 (1)	6.2295 (1)
$9.5694 \cdot 10^{-3}$	7.0059	7.2627 (1)	7.2626 (1)
$1.0000 \cdot 10^{-2}$	8.0088	8.2853 (1)	8.2851 (1)
$1.0417 \cdot 10^{-2}$	9.0088	9.3632 (1)	9.3630 (1)

4. Bubnov–Galerkin Method. It is assumed that in the rectangular plate flutter problem (in the traditional formulation $V = \{v_x, 0\}$) the Bubnov–Galerkin method gives a reasonable value of the critical velocity even in the two-term approximation. However, in [2], it is noted that in the case of a streamwise elongated plate, the effectiveness of the method decreases sharply and to reach reasonable accuracy, it is necessary to retain a considerable (generally speaking, unknown beforehand) number of terms in the approximating sum. The applicability of the Bubnov–Galerkin method to plate flutter problems in a general formulation has been studied insufficiently. The results given below fill in this gap to some extent.

From Figs. 1 and 2, it follows that the characteristic dimension of the perturbation is on the order of half the smaller side of the plate; therefore, an approximate solution was sought in the form $\varphi = c_{mn} \sin(m\pi y) \sin(n\beta\pi x)$, $m = 1, 2$; $n = 1, \ldots, 4$ (the plate occupies the region $K = \{(x, y)\colon\ 0 \leqslant x \leqslant 1/\beta,\ 0 \leqslant y \leqslant 1\}$). The standard procedure of the Bubnov–Galerkin method for solving Eqs. (1.1) reduces to examination of the roots of the characteristic determinant of the eighth order, which is not written here because of its cumbersome form. It is necessary to determine the dependence $\lambda = \lambda(v, \theta)$. An analysis of the results allows the following conclusions to be drawn: (a) the Bubnov–Galerkin method gives satisfactory estimated values of $v^*_{\rm cr}$ if the number of terms in the formula for φ is not less than $N \sim 4a/b$ (two "half-waves" along the smaller side and $2a/b$ "half-waves" along the larger side); (b) in the determination of the dependence $\lambda = \lambda(v, \theta)$, and consequently, the oscillation modes, the Bubnov–Galerkin method gives an error that increases with increasing flow velocity and leads to deterioration of qualitative results. These conclusions are not final; investigation for plates of different geometries and different combinations of boundary conditions is required.

5. Comparison with the Results of [14]. In [14], the critical flutter velocity for a simply supported square plate were calculated in a partial formulation with the flow velocity vector directed parallel to the plate side. In the paper cited, the occurrence of a complex eigenvalue in the spectral problem was examined. In [14], the velocity increment at which the eigenvalue enters a stability parabola was not determined. Below, the results of calculation using the procedure described are given and compared with the results of [14]. The calculations were performed for the following parameter values: $p_0 = 1.0126 \cdot 10^5$ Pa, $\rho_0 = 1.2283$ kg/m^3, $\mu = 0.3$, $k = 1.4$, $E = 1.9982 \cdot 10^{11}$ Pa, and $\rho = 7.8 \cdot 10^3$ kg/m^3. The plate thickness was varied. The calculation results are presented in Table 3. The relative thicknesses of a square plate ($b = 1$) are given in the first column. The values of $v_{\rm cr}$ obtained in [14] are given in the second column. The third and fourth columns give values of $v_{\rm cr}$ calculated using the procedure of Sec. 2, on 9×9 and 19×19 grids, respectively. The eigenvalue number used for stability calculation is given in parentheses. From Table 3, it follows that the results are in good agreement. The largest relative error of 3.8% takes place for $h = 1.0417 \cdot 10^{-2}$.

6. Conclusions. The error of the numerical method described here can be estimated by a standard method. We only note the property of the discretization used. In the present study, the solution was interpolated using Lagrange interpolation polynomials. It is known that this the approximation of the function by the polynomial is the better the larger the number of smoothness conditions to which the function satisfies [15]. It is also known that elliptical equations have solutions of high smoothness (for a rectangle, this is valid inside the region rather than in the corners on the boundary). The calculations performed confirm the high quality of the algorithm. Even on a grid of $9 \times 9 = 81$ at $a/b = 0.5$, the critical velocity is accurate to four decimal places, as follows from comparison with calculations on a grid of $19 \times 19 = 361$. The critical velocity can be determined more accurately (the root of the corresponding transcendental equation was accurate to $\varepsilon = 10^{-4}$) but in practice this is not required.

This work was supported by the Russian Foundation for Fundamental Research (Grant No. 97-01-00923).

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